Soil Plasticity and the Structured Cam Clay Model

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I: Soil plasticity

(1) Knowledge and wisdom from past achievements

The earth, our Mother Earth, made up of rocks and soils.

The area of interest of geotechnical engineering

The Mother Earth is the ground which supports us not only physically but mentally as well, because the extremities of life are associated with the earth. The Earth preserves the wonders of the past and passes down legends and provides us with a beautiful land full of promise. The activity of human beings on earth plays the key role to man's evolution.

Our ancestors' knowledge and wisdom may shock our modern soil engineers and academia. Three examples are given here.

The Pagoda of Phra Pathom Chedi, Thailand, the Great Wall of China, a British merchant.

The Pagoda of Phra Pathom Chedi, Thailand

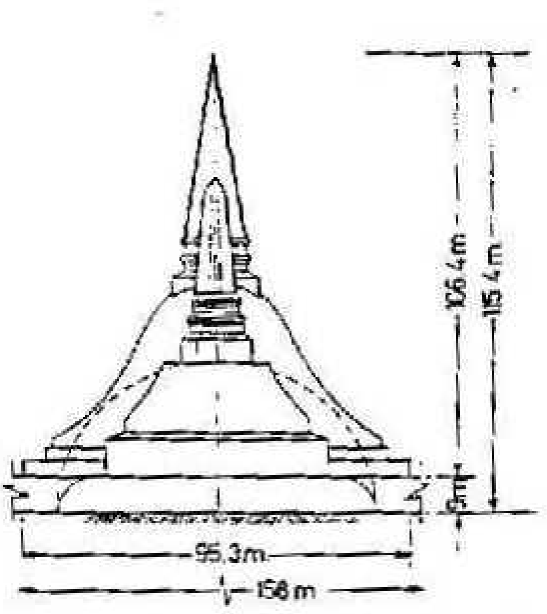


Fig. 1 The Pagoda of Phra Pathom Chedi, Thailand

Built in 300 BC; Height: 115 m; diameter of the circular base: 158 m.

Weight: 500,000 ton.

During the history of over 2300 years, the pagoda stands well, and a highly uniform settlement of 2.5 m has accumulated.

An engineering miracle, more than the Pisa Tower.

The load concentration of the pagoda: 10 MN/m2

Modern design code: 4-5 MN/M2 for high building;

1.5 – 4 MN/M2 for large silos.

Knowledge: (1) it is the non-uniform settlement, not the uniform settlement, that destroys our structures. (2) The bearing capacity of the ground increases with settlement.

Modern soil mechanics with appropriate constitutive models and advanced numerical techniques gives the same conclusion. But modern engineers dare not design such a building.

The Great Wall of China

Fig. 2 The Great Wall of China

Built around 250 BC as military defense project and was over 10,000 km long. With good maintenance after that, the Wall played an important role as National Wall Defense (NWD) until the middle of the 20th century. Its military function lasted over 2,000 years!

Techniques used in the construction of the Wall

(1) Using wood hammers to densify the soil

They knew the strength and stiffness of soil can be effectively improved by increasing the density of soils.

This was an important research topic in 1960s. Even now, you may frequently read research reports on this topic.

(2) Using grass and tree to reinforce soil

They knew how to improve the strength of soil and stiffness of deformation by geo-textile reinforcement.

(3) Using sticky rice to cement gravels, bricks and rocks.

Cementing gravels and rocks to give strong support

Reinforcement by strong materials and by cementation: very popular methods of soil improvement in our modern world since 1970.

Only since 1980, research started on the improvement of soil strength and stiffness by reinforcement and cementation, and we find it is a very challenging topic and much more research is required.

The work in Thailand on this topic is world class (e.g., Prof. Bergado at Asian Institute of Technology, Dr. Horpibulsuk at Suranaree University of Technology).

A British merchant

Darwin (1883) and Renald (1887) made an important discovery of soil property.

When sheared, loose soil will shrink, but a dense soil will expands.

This characteristic of soils, called as dilatancy, is a feature fundamental different from other engineering materials. Rowe (1962) performed a “pioneering” study on dilatancy and proposed the Rowe’s dilatancy law, which marked a milestone in modern soil mechanics.

Nevertheless, Darwin (1883) and Renald (1887) noticed that “Corn merchants have known dilatancy a very long time ago”.

At that time in Britain, Corns were bought and sold by volume, not by weight. A common practice for a buyer would be:

If you buy corns, you shake it.

If corns decrease in volume, they are loose. You shake them until they shrinks no more. If the volume increases, the corns are dense. You leave them alone, you do not touch them.

A good business sense, isn’t it?

Conclusion

Our knowledge of soil mechanics comes from human practice. Our most wonderful discovery in academic world sometimes may be merely a common sense for practitioners.

Our research on soil mechanics is to help human practice. Only by serving human need, soil mechanics is a live theory.

2: Constitutive models and their importance

What is a constitutive model?

> **A constitutive model describes the change in the strain state of an element of material to the change in the stress state acting on the element.**

Mathematically, it can be expressed as

Strain increment **Stress increment**

$$d\varepsilon = f\left(\xi, \chi, \gamma, \sigma', d\sigma'\right) \tag{1}$$

Current stress state

Material properties **Strain history**

Stress history

The response of soil to stress acting on it is dependent on factors such as the current stress state, stress history, and strain history.

A constitutive model provides information on the strength and deformation of a material in an infinitesimal element under stress.

In engineering designs and safety check, we are only concerned about two facts of our structures: the strength and deformation of the structure. Only through the knowledge of the strength and deformation of an element of the material can we compute the strength and deformation of an engineering structure. Moreover, now that we have good mathematical models and advanced numerical analysis packages, the results of engineering computation are essentially controlled by the accuracy of our constitutive models and reliable determination of model parameters.

This is the importance and also the reason for constitutive modeling of soils.

3: General Elasticity

Modern soil mechanics was founded on the day when Terzaghi wrote (1936),

$$\sigma' = \sigma - u \tag{2}$$

This is the principle of effective stress, the sole foundation stone of modern soil mechanics. And Terzaghi became the founder of modern soil mechanics.

With the principle of effective stress we can make a sensible link between the deformation of soil and the stress acting on it. Hereafter, principles of continuum mechanics are found to be applicable to soils.

With the effective stress principle, theory of elasticity was naturally introduced into soil mechanics. The law for elastic deformation was firstly proposed by Hook (1678) as "**deformation is proportional to force**".

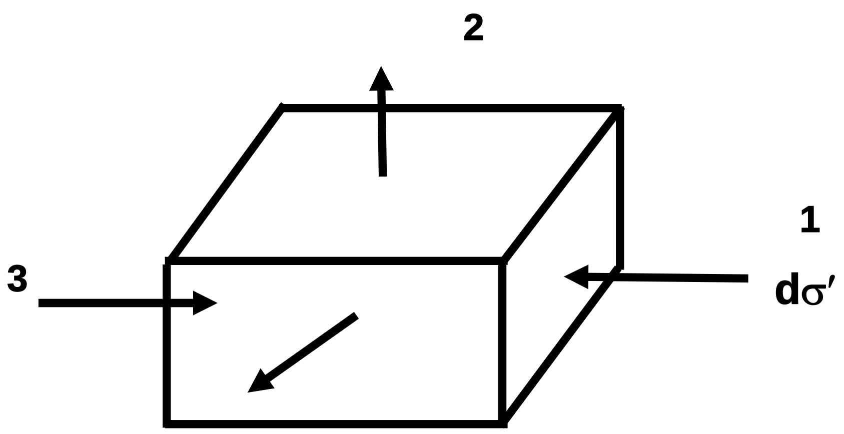


Fig. 3 Deformation of a soil element under stress

At the element level (Fig. 3), Hook's law can be written as:

Strain is proportional to stress.

For soil, the following incremental form is suggested,

Stiffness, a material constant, or the Young's modulus

$$d\varepsilon = \frac{d\sigma'}{E} \tag{3}$$

The strain and the stress are in one direction, i.e., along direction 1.

Elastic response of soil to loading is dependent on mechanical property of the soil, and that mechanical property of the soil is represented by a material constant, the Young's modulus.

Later, it was observed that deformation can also be found at direction vertical to the applied stress, i.e., in the directions 2 and 3. Hence, a complete Hook's law for stress $d\sigma'_1$ is obtain as

$$d\varepsilon_1 = \frac{d\sigma'_1}{E_1} \qquad d\varepsilon_2 = -\nu_{12}\frac{d\sigma'_1}{E_1} \qquad d\varepsilon_3 = -\nu_{13}\frac{d\sigma'_1}{E_1} \tag{4}$$

Young's modulus E in direction 1

Poisson's ratio ν for deformation in direction 2 by stressing in direction 1

Poisson's ratio ν for deformation in direction 3 by stressing in direction 1

The sign of the strain for the Poisson's effective is negative. This means that expansive deformation will be induced in direction vertical to the applied stress if the stress induces directly compression.

For linear elastic problems with small deformation, the deformation of soil corresponding to the increments of the three principal stresses $d\sigma'_1$, $d\sigma'_2$, and $d\sigma'_3$, can be obtained by superimposing the individual deformation. Therefore

$$\begin{cases} d\varepsilon_1 = \dfrac{d\sigma'_1}{E_1} - \nu_{21}\dfrac{d\sigma'_2}{E_2} - \nu_{31}\dfrac{d\sigma'_3}{E_3} \\ d\varepsilon_2 = \nu_{12}\dfrac{d\sigma'_1}{E_1} - \dfrac{d\sigma'_2}{E_2} - \nu_{32}\dfrac{d\sigma'_3}{E_3} \\ d\varepsilon_3 = \nu_{13}\dfrac{d\sigma'_1}{E_1} - \nu_{23}\dfrac{d\sigma'_2}{E_2} - \dfrac{d\sigma'_3}{E_3} \end{cases} \tag{5}$$

Via the principle of virtual work, we know $\nu_{12}= \nu_{21}$, $\nu_{23}= \nu_{32}$, $\nu_{13}= \nu_{31}$.

The above equation is a general anisotropic elastici equation, because

(1) Young's modulus in direction 1, E1, can be different from E2 and E3, and

(2) Poisson's ratio for ν12, deformation in direction 2 by loading in direction 1, can be different from ν23 and ν31.

For an isotropic material, E1=E2=E3=E, ν12= ν21= ν23= ν32= ν13= ν31=ν. Then, the isotropic elastic equation is

$$\begin{cases} d\varepsilon_1 = \dfrac{d\sigma'_1}{E} - \nu\dfrac{d\sigma'_2}{E} - \nu\dfrac{d\sigma'_3}{E} \\ d\varepsilon_2 = \nu\dfrac{d\sigma'_1}{E} - \dfrac{d\sigma'_2}{E} - \nu\dfrac{d\sigma'_3}{E} \\ d\varepsilon_3 = \nu\dfrac{d\sigma'_1}{E} - \nu\dfrac{d\sigma'_2}{E} - \dfrac{d\sigma'_3}{E} \end{cases} \qquad \textbf{(6)}$$

Elastic deformation is a deformation completely recoverable. Thus elastic deformation is dependent on the value of the stress, but independent of the stress path, e.g., how soil is loaded to the stress state. Mathematically, it can be expressed as

$$\oint d\varepsilon = 0 \qquad (7)$$

There is no change in elastic deformation for loading along an enclosed stress path.

4: Original Cam Clay model and soil plasticity

The proposal of the original Cam Clay model proposed by Roscoe, Schofield and Wroth (1958), is a revolutional development in modern soil mechanics. Mechanical properties of soil have been unified elegantly and consistently into the model and Soil Mechanics hereafter is a systematic science.

4.1: Mathematical model

When facing an object too large or too complicated for us to handle, we form a model for it by idealization, simplification and approximation. A model is a human invention for understanding and analyzing an object. A model represents the features of an object of first importance. Matters of second importance are ignored, and matters of great complexity for details are often simplified.

The original Cam Clay model was formed by studying the deformation of soils in laboratory reconstituted states, not the states you find in nature. As a result, the model is only suitable for describing the behaviour of soils in laboratory reconstituted states.

Reconstituted clay: soils taken from nature, drying and breaking to powder, then mixing with water carefully.

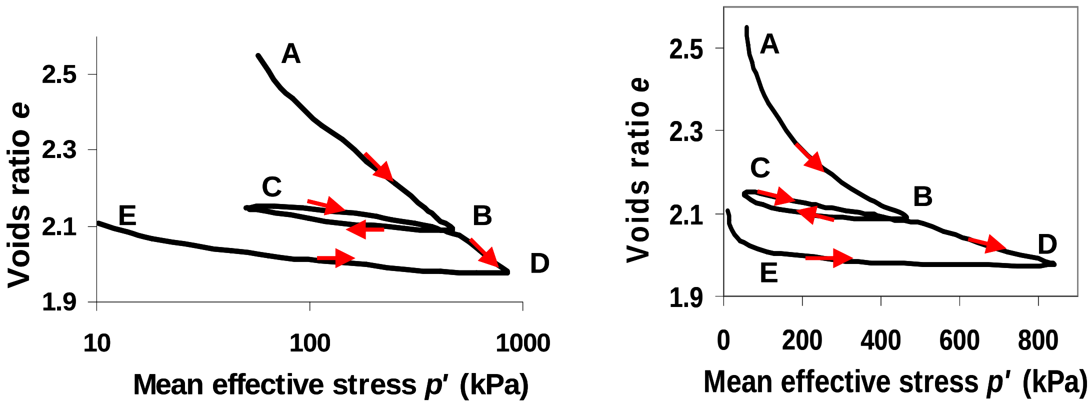


Fig. 4 Compression behaviour of reconstituted kaolin clay during an isotropic test

4.2: A model for the compression behaviour of soil

How to model the deformation of soil? Where should we start?

We start by examining the behaviour of soil observed in experiments, and start from the simplest situations possible. Let's examine experimental data on the isotropic compression test on reconstituted Kaolin clay. For an isotropic compression test, $\sigma'_1=\sigma'_2=\sigma'_3$. The mean effective stress p' is given by

$$p' = \frac{1}{3}(\sigma_1' + \sigma_2' + \sigma_3') = \sigma_1' \tag{8}$$

The results are presented in two scales, *e*-*p*′ scale and *e*-ln*p*′ (Fig. 4). The stress path is: loading from A to B, unloading from B to C, then reloading from C to B and to D, and then unloading again from D to E.

The compression behaviour of soil in the *e* – ln*p*′ space is approximately linear. Everyone, whether a scientist or an engineer, loves linearality. So soil behaviour is studied in the *e* – ln*p*′ space.

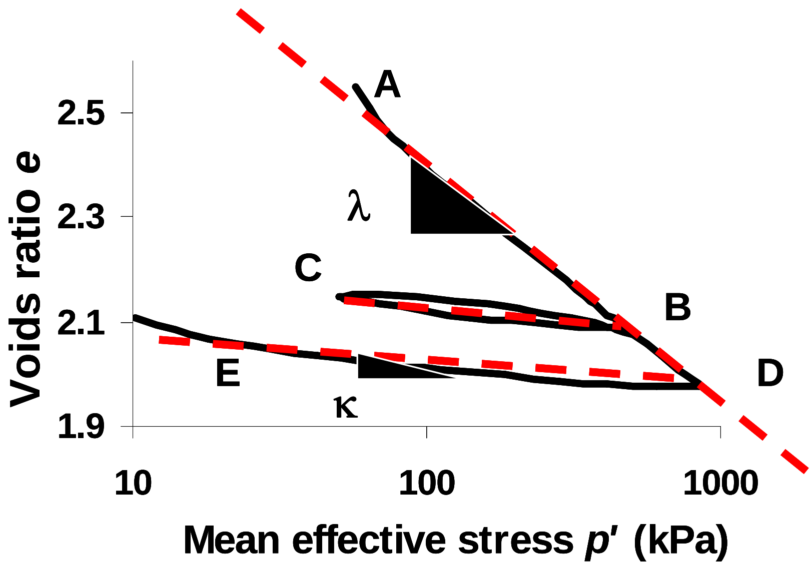


Fig. 5 Compression behaviour of reconstituted kaolin clay in *e* – ln*p*′ space

We can see that for loading along AB abd BD, the variation of the voids ratio *e* with the mean effective stress ln*p*′ is obviously in one line.

What type of loading is it? During the loading, the current mean effective stress is the maximum mean effective stress the soil has ever experienced. It is **virgin loading**. Therefore, the first assumption we make for the compression model is as follows.

For virgin loading, the variation of the voids ratio *e* with the mean effective stress ln*p*′ is linear.

The gradient of the compression line the *e* – ln*p*′ line is denoted by λ, the virgin compression index (Fig. 5).

Let's examine soil behaviour along B➔C➔B, D➔E. What type of loadings are they? During these loadings, the current mean effective stress is less than the historical maximum stress the soil experienced. B➔C and D➔E are unloading, and C➔B is reloading.

In unloading, the voids ratio increases with the reduction in mean effective stress, and in reloading, the voids ratio decreases with the increase in mean effective stress. There is a small hysterestic loop. Thus, soil deformation during unloading and reloading is not completely recoverable. If the hysterestic loop is ignored, soil deformation during unloading and reloading can be approximately treated as one line, thus the deformation during unloading and reloading is completely recoverable. Then soil deformation can be assumed as elastic. Also it can be seen that the variation of the voids ratio e with the mean effective stress p' during unloading and reloading can be simplified as linear. Therefore, the second assumption we make for the compression model is as follows.

During unloading and reloading, the voids ratio *e* varies elastically and linearly with the mean effective stress ln*p'*.

The elastic deformation of soil can be described by Hook's law. The gradient of the compression line the $e - \ln p'$ line is denoted by κ, the swelling and recompression index (Fig. 5)

4.3: A number of important concept about plastic defromation

4.3.1: Elastic deformation and plastic deformation

The deformation of soil can be divided into elastic deformation and plastic deformation, thus

$$d\varepsilon = d\varepsilon^{e} + d\varepsilon^{p} \tag{9}$$

The plastic deformation is not recoverable, e.g., the deformation produced by applying a force does not diminish when the force is removed.

4.3.2 : Yielding of soil

Let's examine the compression behaviour of reloading of the kaolin clay in Fig. 5. For loading from CBD, at point B, there is a sharp change in the stress and strain curve. Point B is the state where the current stress state reaches the maximum stress the soil has ever experienced. For further loading, the current mean effective stress will be the maximum mean effective stress the soil has ever experienced. It is virgin loading. Plastic deformation takes places at point B.

At point B, soil behaviour changes from pure elastic behaviour to plastic behaviour, and there is a sharp change in the stress and strain curve. We say yielding occurs at B. Consequently, the yielding points can be identified experimentally by observing the sharp change of the stress and strain curve.

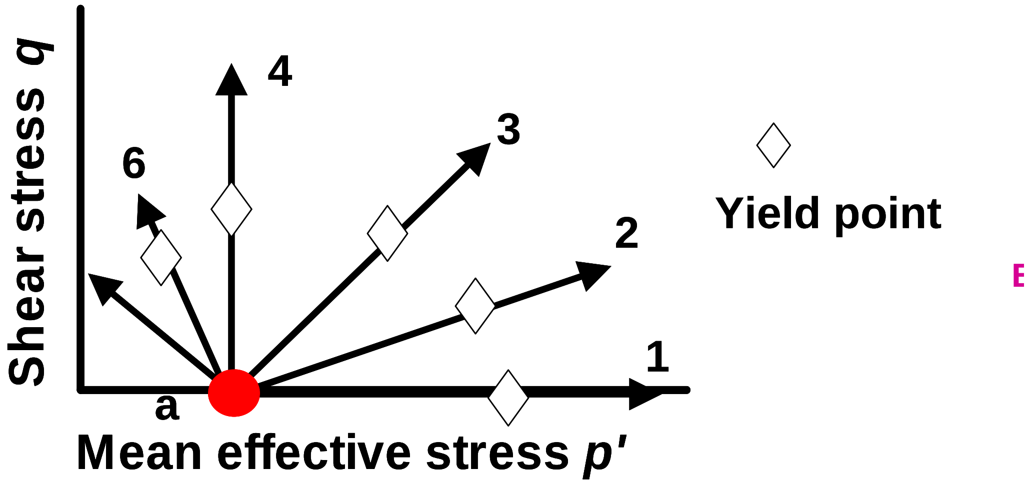


Fig. 6 Yield surface for soil

4.3.3: Yield surface for soils

For loading along stress path a1, i.e., isotropic compression, there is a yielding point (Fig. 6), marked by open diamond. For all other stress paths, such as a2, a3, a4, , Soil behaviour is similar to that for isotropic compression, a yielding point is found for every stress paths.

All the yielding points make up a boundary in the p'-q space (Fig. 6). This boundary is a yield surface for the soil.

Similar to the idealization of the compression behaviour, soil behaviour is divided into two regions by the yield surface: the pure elastic deformation region and plastic deformation regions. (1) Loading inside the yield surface, pure elastic deformation takes place. (2) Loading on the yield surface and causing it expansion, plastic deformation takes place.

If a soil has no cohesion, e.g., a non cohesive soil, tensile force cannot be applied. Then the yield surface is valid only for $p' \geq 0$. Because soil is a frictional material, yield surface for non-cohesive soils passes through the origin of the stress coordinates, the stress state (p'=0, q=0).

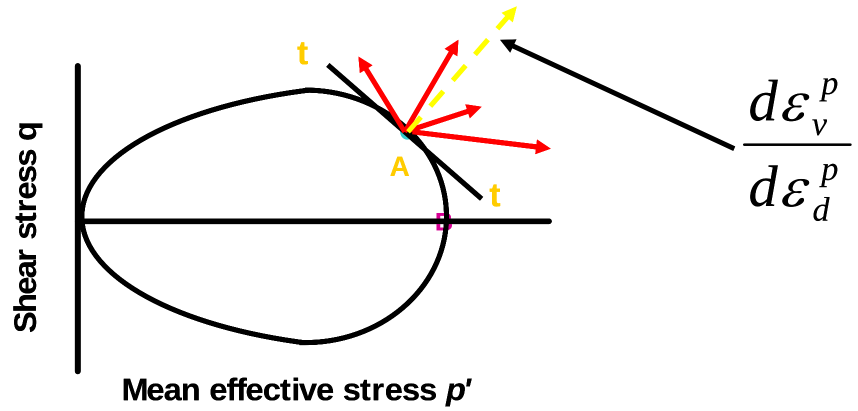


Fig.7 Flow rule or the direction of the plastic strain increment

4.3.4: Flow rule or plastic potential

A flow rule defines the direction of plastic strain increment, i.e.,

$$\frac{d\varepsilon_v^p}{d\varepsilon_d^p}$$

At any stress state on the yield surface, plastic deformation occurs for all the loadings pointing outside the yield surface, i.e., as shown in Fig. 7.

A conlusion in the study of metal plasticity is that the direction of plastic strain increment is dependent on the stress state, independent of stress increment. It is comfirmed by experimental observation that the same rule is applicable for soils.

Thus for all the loadings at A, which induce plastic deformation, the direction of the plastic strain increment is the same and can be expressed in terms of the current stress state, i.e.,

$$\frac{d\varepsilon_v^p}{d\varepsilon_d^p} = r(p', q) \qquad \textbf{(10)}$$

A plastic potential $g(p',q)$ is in the stress space, and the normal of which gives the direction of the plastic strain increment vector, i.e.,

$$\frac{\partial g / \partial p'}{\partial g / \partial q} = \frac{d\varepsilon_v^p}{d\varepsilon_d^p} \qquad \textbf{(11)}$$

4.5: Yield surface and plastic potential for original Cam Clay model

4.5.1: Equation for the dissipation of energy

The yield surface and plastic potential for original Cam Clay model was derived from a hypothesis of energy dissipation. The concept of energy dissipation is a very importance concept, a fundamental natural law. This law is in a higher order than constitutive relations, even principles of continuum mechanics. Roscoe, Schofield and Wroth (1958) proposed the following hypothesis of energy dissipation for soil.

> **Soil is a frictional material and the dissipation of the plastic energy is proportional to distortional strain increment and the mean effective stress acting on it.**

Plastic work input to an element of soil can written as:

$$dW = p'd\varepsilon_v^p + qd\varepsilon_d^p \qquad \textbf{(12)}$$

Thus the following energy dissipation equation can be obtained,

$$p'd\varepsilon_v^p + qd\varepsilon_d^p = \mathrm{M}p'd\varepsilon_d^p \qquad \textbf{(13)}$$

a thermodynamic material constant defining the rate of energy dissipation

4.5.2: Drucker's stability criterion and normality

Drucker proposed the following stability criterion (1952)

The work done by the external agency on the change in displacement it produces must be positive or zero

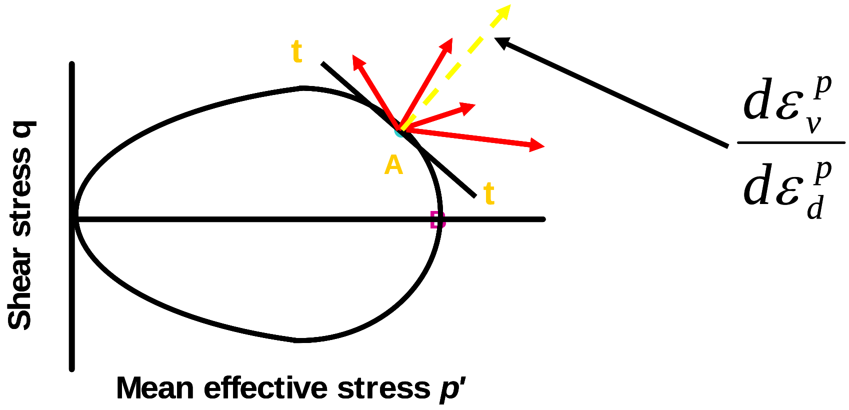


Fig.8 Drucker's stability and normality

For an element of soil that is in equilibrium, the original stress and strain state are denoted as (p', q; ε_v, ε_d). An additional force (dp', dq) is applied , and the displacement it produces is ($d\varepsilon_v$, $d\varepsilon_d$). Mathematically, for an element of soil, the Drucker's criterion can be written as

$$\int \left(dp'd\varepsilon_v + dqd\varepsilon_d\right) \geq 0 \tag{14}$$

For an enclosed stress path, we obtain

$$\oint \left(dp'd\varepsilon_v + dqd\varepsilon_d\right) \geq 0 \tag{15}$$

The total strain increment is made up of elastic and plastic parts, i.e., equation (9). Considering that the work done by elastic deformation in an enclosed stress path is zero, then we obtain

$$\oint \left(dp'd\varepsilon_v^p + dqd\varepsilon_v^p\right) \geq 0 \tag{16}$$

Now we know that (1) for all stress increments that point outside the yield surface plastic deformation will be induced, and that (2) there is one unique direction for the plastic strain increment. For all stress paths, Drucker's stability criterion must be satisfied. Then, the plastic strain vector is necessarily normal to the yield surface. This is what we call normality.

Plastic strain increment is normal to the yield surface.

If the plastic strain increment vector is normal to the yield surface, the soil has **associated flow rule**.

If the plastic strain increment vector is not normal to the yield surface, the soil has **non-associated flow rule**.

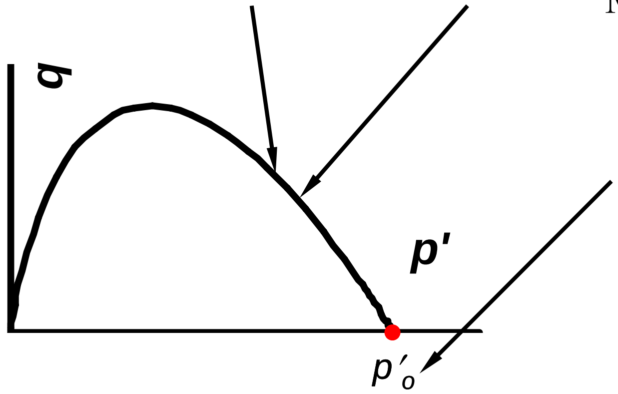
For soils with associated flow rule, the plastic potential is identical to the yield surface. Therefore,

$$g\left(p', q, p'_o\right) = f\left(p', q, p'_o\right) \tag{17}$$

4.5.3 Yield surface and plastic potential

As soil has associated flow rule, the plastic potential and the yield surface are identical. The partial differential equation given by the energy dissipation equation can be solved. We obtain the yield surface and plastic potential for the original Cam Clay model as follows,

$$f\left(p', q, p'_c\right) = g\left(p', q, p'_c\right) = \frac{q}{\mathrm{M}p'} - \ln\left(\frac{p'_o}{p'}\right) = 0 \tag{18}$$



p'_o: size of the yield surface
For isotropic loading, $\sigma'_1 = \sigma'_1 = \sigma'_1$, then $p' = p'_o$.

Fig.9 Yield surface and plastic potential for soils

Stress ratio η is defined as

$$\eta = \frac{q}{p'} \tag{19}$$

Then the yield surface can also be written as

$$f\left(p', q, p'_c\right) = g\left(p', q, p'_c\right) = \frac{\eta}{\mathrm{M}} - \ln\left(\frac{p'_o}{p'}\right) = 0 \tag{20}$$

4.5.4 Flow rule

Based on the function for the yield surface, the following flow rule is obtained

$$\frac{d\varepsilon_v^p}{d\varepsilon_d^p} = \frac{\partial g / \partial p'}{\partial g / \partial q} = \mathrm{M} - \eta \qquad \textbf{(21)}$$

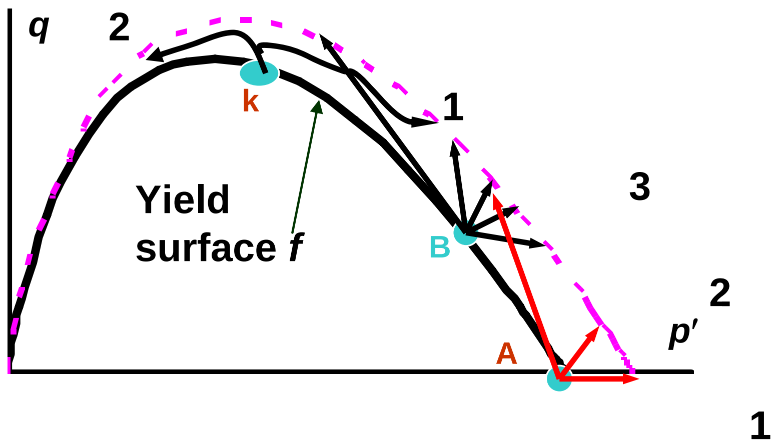


Fig. 10 Size change of the yield surface

4.6: Hardening of soil

Hardening of yield surface is described as the expansion of the yield surface with plastic deformation. The shrinkage of the yield surface such as that occurring during softening or instability may be considered as negative hardening.

A fundamental contribution to soil plasticity by Roscoe, Schofield and Wroth (1958) in the formulation of the Cam Clay model is about the hardening law of soils. It is proposed as

> **The hardening of yield surface is dependent on the plastic volumetric deformation only.**

This is the volumetric hardening assumption. There are the following Consequences:

(1) There is one to one relationship between size of the yield surface and the plastic volumetric deformation.

(2) The plastic volumetric deformation is determined by the change in size of the yield surface only, irrespective of loading stress paths.

Consequently, the plastic volumetric deformation can be linked to the size change of the yield surface as follows

$$d\varepsilon_v^p = \Lambda dp_o' \tag{22}$$

Since the flow rule is given by eq. (21), we can determine all the plastic deformation if we find out the equation describing the relationship between the size change of the yield surface and the plastic volumetric deformation.

There are three stress increments at stress state A with the size of the current yield surface being p'_o (Fig. 10). All the three loadings result in the same change of the yield surface, dp'_o. As a result, the plastic volumetric deformation induced by the three paths is the same. Indeed all the loadings that result the same change in the yield surface, no matter where the stress state sits on the yield surface, will produce the same plastic volumetric deformation, such as stress paths at stress states A, B and K.

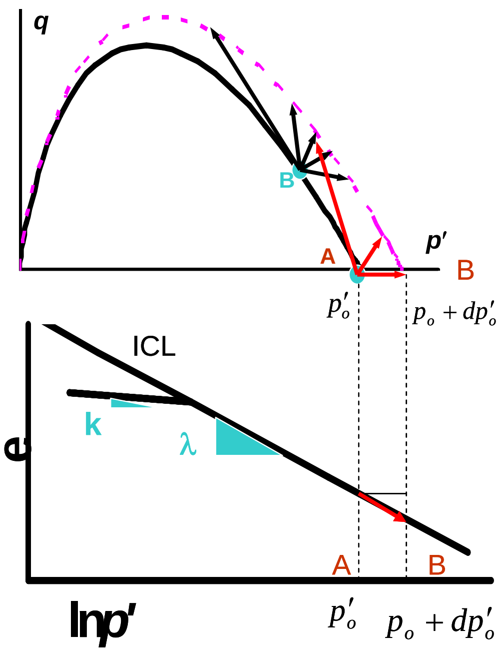


Fig. 11 Hardening of soil

We have already formed a compression model for soil, and the elastic and plastic volumetric deformation during isotropic loading is well studied. So we try to establish the relationship between dp'_o and $d\varepsilon_v^p$ from studying soil behaviour during an isotropic compression test.

As shown in Fig. 11, soil behaviour for isotropic tests can be described by the Isotropic Compresses Line (ICL). For virgin loading, where plastic deformation occurs, the gradient for ICL is λ, the virgin compression index. For unloading and reloading, soil behaves elastically. The compression index for elastic deformation is k.

Let's examine soil behaviour for virgin loading from A to B. At A, soil state is described by (p'_A, e_A). A stress increment dp' is given, and soil state changes to B with(p'_A+ dp', e_A+de).

For loading from stress state A to B, soil state in the $e - \ln p'$ plane travels along ICL. The reduction in voids ratio is,

$$de = -\frac{\lambda}{p_A} dp' \tag{23}$$

The corresponding total volumetric strain increment can be calculated as

$$d\varepsilon_v = -\frac{de}{1+e} = \frac{\lambda}{(1+e)p_A} dp' \tag{24}$$

Elastic volumetric strain increment can be worked out by Hook's law as follows

$$d\varepsilon_v^e = \frac{\kappa}{(1+e)p_A} dp' \tag{25}$$

The difference between the total volumetric deformation and the elastic volumetric deformation gives the plastic deformation. Thus

$$d\varepsilon_v^p = d\varepsilon_v - d\varepsilon_v^e = \frac{(\lambda-\kappa)}{(1+e)p_A} dp' \tag{26}$$

The plastic volumetric deformation is dependent on the size change of the yield surface only. Thus equation (26) has to be rearranged in terms of the size of the yield surface, not magnitudes of the current stress. For isotropic loading, we know

$$p'_o=p' \text{ , and } dp'_o=dp'$$

Thus, equation (26) can be written as

$$d\varepsilon_v^p = \frac{(\lambda-\kappa)}{(1+e)p_o} dp_o' \quad \text{for } dp_o' > 0 \tag{27}$$

Only loading on the yield surface which results in the expansion of the surface does plastic deformation occur; thus a condition $dp'_o > 0$ is imposed. A relationship between $d\varepsilon_v^p$ and the change of yield surface for isotropic compression is thus obtained. Because the hardening of yield surface is dependent on the plastic volumetric deformation only. The above equation **is valid for all stress increments that result in the same change in the yield surface**

4.7: Plastic deformation and the incremental stress and strain relationship

With all the work performed, the plastic deformation can be worked out and the original Cam Clay model is defined. Let's summarize the work introduced in Sections from 4.3 to 4.6.

Yield surface

The yield surface is worked out from the energy dissipation function and Drucker's stability criterion. It is given as

$$f(p', q, p_o') = \frac{q}{\mathrm{M}p'} - \ln\left(\frac{p_o'}{p'}\right) = 0 \tag{28}$$

The size of the yield surface p'_o can be determined from any stress state on the surface as follows

$$p_o' = p' \exp\left(\frac{q}{\mathrm{M}p'}\right) = p' \exp\left(\frac{\eta}{\mathrm{M}}\right) \tag{29}$$

The increment of the yield surface in size corresponding to a stress increment (dp', dq) can be worked out as

$$dp_o' = p_o'\left\{\left[1 - \ln\left(\frac{p_o'}{p'}\right)\right]\frac{dp'}{p'} + \frac{dq}{\mathrm{M}p'}\right\} \tag{30}$$

Flow rule

Soil is assumed to have associated flow rule, then

$$\frac{d\varepsilon_d^p}{d\varepsilon_v^p} = \frac{\partial f \big/ \partial p'}{\partial f \big/ \partial q} = \mathrm{M} - \eta \tag{31}$$

Hardening of the yield surface

The hardening of soil is dependent on plastic volumetric deformation only. The relationship between the hardening of the yield surface and plastic volumetric deformation is

$$d\varepsilon_v^p = \frac{(\lambda-\kappa)}{(1+e)p_o}dp_o' \tag{32}$$

Consequently, the plastic deformation can be worked out. Suppose a stress state (p', q) on the yield surface. When there is a stress increment (dp', dq), which results in the expansion of the yield surface, plastic deformation will be induced. The size change of the yield surface can be worked by eq. (30), then the plastic volumetric deformation can be worked out by eq. (32) and the plastic distortional deformation can be obtained by eq. (31), therefore,

$$\begin{cases} d\varepsilon_v^p = \dfrac{(\lambda-\kappa)}{(1+e)}\dfrac{dp_o'}{p_o'} = \dfrac{(\lambda-\kappa)}{(1+e)}\left\{\left[1-\ln\left(\dfrac{p_o'}{p'}\right)\right]\dfrac{dp'}{p'}+\dfrac{dq}{\mathrm{M}p'}\right\} \\ d\varepsilon_d^p = \dfrac{d\varepsilon_v^p}{(\mathrm{M}-\eta)} = \dfrac{(\lambda-\kappa)}{(1+e)(\mathrm{M}-\eta)}\left\{\left[1-\ln\left(\dfrac{p_o'}{p'}\right)\right]\dfrac{dp'}{p'}+\dfrac{dq}{\mathrm{M}p'}\right\} \end{cases} \quad \text{for } dp_o' > 0 \tag{33}$$

Plus the elastic deformation the following incremental stress and strain relationship is obtained

$$\begin{cases} d\varepsilon_v = \dfrac{\kappa}{(1+e)}\dfrac{dp'}{p'} + \dfrac{(\lambda-\kappa)}{(1+e)}\left\{\left[1-\ln\left(\dfrac{p_o'}{p'}\right)\right]\dfrac{dp'}{p'}+\dfrac{dq}{\mathrm{M}p'}\right\} \\ d\varepsilon_d = \dfrac{2\kappa(1+\nu)}{9(1+e)(1-2\nu)}\dfrac{dq}{p'} + \dfrac{(\lambda-\kappa)}{(1+e)(\mathrm{M}-\eta)}\left\{\left[1-\ln\left(\dfrac{p_o'}{p'}\right)\right]\dfrac{dp'}{p'}+\dfrac{dq}{\mathrm{M}p'}\right\} \end{cases} \quad \text{for } dp_o' > 0 \tag{34a}$$

And

$$\begin{cases} d\varepsilon_v = \dfrac{\kappa}{(1+e)}\dfrac{dp'}{p'} \\ d\varepsilon_d = \dfrac{2\kappa(1+\nu)}{9(1+e)(1-2\nu)}\dfrac{dq}{p'} \end{cases} \quad \text{for } dp_o' < 0 \tag{34b}$$

The change in the strain state of an element of material resulting from the change in the stress state acting on the element is defined, and the original Cam Clay model is completed.

4.8: Two basic features of the original Cam Clay model

The Cam Clay model successfully unifies consistently the mechanical properties of soil into a simple and elegant theoretical framework. The formulation of the Cam Clay model is the most important development in modern soil mechanics.

Two most distinguished features of the original Cam Clay model are discussed here. They are (1) the plastic volumetric dependent of hardening of soils, and (2) the existence of a critical state of deformation as the final failure state. These two features can be seen as the marks of models of the Cam Clay family.

(1) Volumetric dependent of hardening

The plastic-volumetric-deformation-dependent hardening of soil is proposed as follows

> **The hardening of yield surface is dependent on the plastic volumetric deformation only.**

Consequently, plastic volumetric deformation is uniquely determined by the change of the yield surface in size; and there is one to one relationship between the size of the yield surface and plastic volumetric deformation. All stress states which have the same accumulation of plastic volumetric strain constitute a single yield surface. Because the elastic volumetric deformation can be calculated from the current stress state, the value of the voids ratio minus the elastic contribution thus uniquely defines the plastic volumetric strain. Therefore, the size of the yield surface is related to the current voids ratio, and the current stress state from which the elastic volumetric deformation is computed.

(2) Prediction of the existence of a critical state of deformation

Let's examine the flow rule and the energy dissipation function

$$\frac{d\varepsilon_v^p}{d\varepsilon_d^p} = \mathrm{M} - \eta$$

$$p' d\varepsilon_v^p + q d\varepsilon_d^p = \mathrm{M} p' d\varepsilon_d^p$$

At a special stress state, η=M, from flow rule,

$$\frac{d\varepsilon_v^p}{d\varepsilon_d^p} = \infty$$

Hence, at this special state, soil has no resistance to shear deformation. Like water or gas, the material has no resistance to any further distortion.

Examining the energy dissipation at this special state. For no change of volumetric deformation, i.e., $d\varepsilon_v^p = 0$, any value of $d\varepsilon_d^p$ can satisfy the equation. Therefore,

Soil can remain at the state η=M with no change in its stress, $dp'_o = 0$, and no volumetric plastic deformation. However, the shear plastic deformation can be infinitive. This is a Critical State of deformation.

A Critical State of deformation is defined as

> **At a Critical State of deformation, a soil has no resistance to shear deformation and the soil can be distorted continuously with its stress state and voids ratio remain unchanged.**

A critical state of deformation is a final failure state. Cam Clay model is the first Critical State model. The theoretical framework, unifying consistently the mechanical properties of soil into one simple and elegant system under the original Cam Clay model, is referred to as the Critical State Soil Mechanics (CSSM). With the introduction of the CSSM soil mechanics is established as a cohesive science.

5: Models of the Cam Clay family

To be added.

6: Notation used in this note

For simplicity, stress and strain states of axisymmetrical conditions are considered in this note. σ'_1 (or ε_1) and σ'_3 (or ε_3) are the axial effective stress (strain), and the radial effective stress (strain) respectively. The mean effective stress p', shear stress q and stress ratio η are given by

$$p' = \frac{1}{3}\left(\sigma_1' + 2\sigma_3'\right)$$

$$q = \left(\sigma_1' - \sigma_3'\right)$$

$$\eta = \frac{q}{p'} \ .$$

The corresponding (work-conjugate) volumetric strain increment, $d\varepsilon_{v,}$, and shear strain increment, $d\varepsilon_d$, are defined by

$$d\varepsilon_v = d\varepsilon_1 + 2d\varepsilon_3$$

and

$$d\varepsilon_d = \frac{2}{3}\left(d\varepsilon_1 - d\varepsilon_3\right)$$

e, voids ratio of a soil; ν, Poisson's ratio,

E, Young's modulus; κ, elastic swelling and recompression index;

λ, compression index for virgin yielding; M, critical state strength.